Algorithms, Games, and Evolution

Theoretical Computer Science Meets other Disciplines

> “Nothing in biology makes sense except in the light of evolution.”
>
> *Theodosius Dobzhansky*

Contribution of this paper

Evolution

=

Multiplicative Weight Update Algorithm

applied to

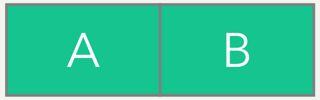
Finding Nash equilibrium of a coordination game

Theoretical Biology

Modelling Evolution

How we model evolution

- Population of an infinite number of members
- Each member has a genotype consisting of two genes



A B

- Two possible alleles per gene

A1 B1

A2 B1

A1 B2

A2 B2

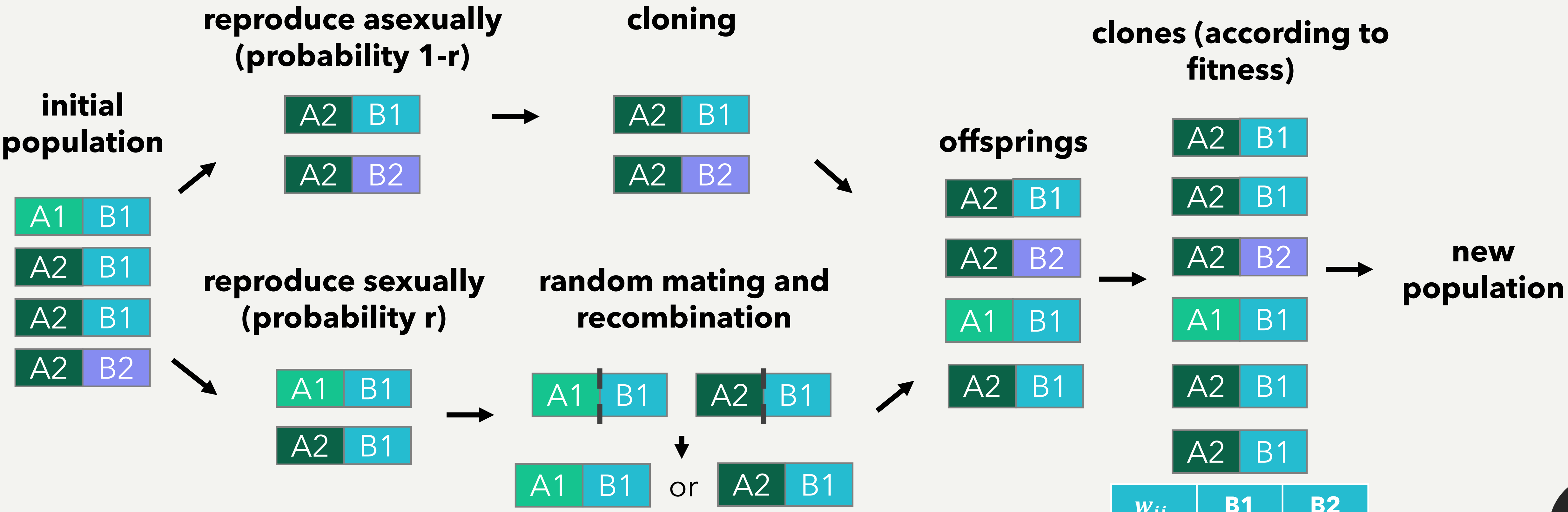
- (Generalisation to an arbitrary number of genes with arbitrarily many alleles is not difficult)

How we model evolution

- fitness value of genotypes

w_{ij}	B1	B2
A1	1	3
A2	2	1

How we model evolution



initial population

A1 B1
A2 B1
A2 B1
A2 B2

reproduce asexually (probability 1-r)

A2 B1
A2 B2

cloning

A2 B1
A2 B2

reproduce sexually (probability r)

A1 B1
A2 B1

random mating and recombination

A1 B1 A2 B1

A1 B1 or A2 B1

offsprings

A2 B1
A2 B2
A1 B1
A2 B1

clones (according to fitness)

A2 B1
A2 B1
A2 B2
A1 B1
A2 B1
A2 B1

new population

w_{ij}	**B1**	**B2**
A1	1	3
A2	2	1

How we model evolution

frequency of
genotype **Ai-Bj** at
time **t+1**

↙

$$p_{ij}^{t+1} = \frac{w_{ij}}{Z^t}\left((1-r)p_{ij}^t + r\left(p_{i1}^t + p_{i2}^t\right)\left(p_{1j}^t + p_{2j}^t\right)\right)$$

How we model evolution

frequency of
genotype **Ai-Bj** at
time **t**

$$p_{ij}^{t+1} = \frac{w_{ij}}{Z^t}\left(\underline{(1-r)p_{ij}^{t}} + r(p_{i1}^{t} + p_{i2}^{t})(p_{1j}^{t} + p_{2j}^{t})\right)$$

asexual
reproduction

How we model evolution

frequency of
genotype **Ai-Bj** at
time **t**

$$p_{ij}^{t+1} = \underline{(1-r)p_{ij}^{t}} + \underline{r\left(p_{i1}^{t} + p_{i2}^{t}\right)\left(p_{1j}^{t} + p_{2j}^{t}\right)}$$

asexual
reproduction

sexual reproduction by
recombination

How we model evolution

$$p_{ij}^{t+1} = \frac{w_{ij}}{Z^t}\left(\underline{(1-r)p_{ij}^t + r\left(p_{i1}^t + p_{i2}^t\right)\left(p_{1j}^t + p_{2j}^t\right)}\right)$$

↑

probability of asexual or sexual reproduction
leading to the genotype **Ai-Bj**

How we model evolution

fitness of genotype **Ai-Bj**
(«number of clones of each offspring with this genotype»)

$$p_{ij}^{t+1} = \frac{w_{ij}}{Z^t}\underline{\left((1-r)p_{ij}^t + r\left(p_{i1}^t + p_{i2}^t\right)\left(p_{1j}^t + p_{2j}^t\right)\right)}$$

probability of asexual or sexual reproduction
leading to the genotype **Ai-Bj**

How we model evolution

fitness of genotype **Ai-Bj**
(«number of clones of each offspring with this genotype»)

$$p_{ij}^{t+1} = \underline{w_{ij}\left((1-r)p_{ij}^{t} + r\left(p_{i1}^{t} + p_{i2}^{t}\right)\left(p_{1j}^{t} + p_{2j}^{t}\right)\right)}$$

Expected number of offsprings with the
genotype **Ai-Bj**

How we model evolution

$$p_{ij}^{t+1} = \frac{w_{ij}}{Z^t}\left((1-r)p_{ij}^t + r\left(p_{i1}^t + p_{i2}^t\right)\left(p_{1j}^t + p_{2j}^t\right)\right)$$

normalization constant such that all frequencies sum up to **1**

Evolution implements MWUA

Evolution

=

Multiplicative Weight Update Algorithm

applied to

Finding Nash equilibrium of a coordination game

Computer Science

Multiplicative Weight Update Algorithm

MWUA

- Simple and general algorithm used in many different areas:

- Machine Learning (e.g. AdaBoost)
- Approximations for NP-hard problems
- Solving linear programs
- **Finding Nash equilibria** for some types of games

Game Theory

MWUA and Coordination Games

Coordination Games

- Two players A and B play a game
- **Reward for player A = reward for player B**

→ They strive to reach the same goal!

Two friends decided to go to a concert together, but they both forgot if they had agreed to see **The Chainsmokers** or **Adele**. They can't communicate until they reach the location.

	The Chainsmokers	**Adele**
The Chainsmokers	2 / 2	0 / 0
Adele	0 / 0	1 / 1

Coordination Games

Two friends decided to go to a concert together, but they both forgot if they had agreed to go to see **The Chainsmokers** or **Adele**. They can't communicate until they reach the location.

	The Chainsmokers	**Adele**
The Chainsmokers	2 / 2	0 / 0
Adele	0 / 0	1 / 1

- How to find the perfect action? (knowing all rewards)
- (Besides the obvious way…)
- They can use the **multiplicative weight update algorithm (MWUA)**

MWUA for games

Given reward function **r** → find the optimal strategies $\mathbf{p_A}$ and $\mathbf{p_B}$

- We start with any strategies $\mathbf{p_A}^{\mathbf{0}}$ and $\mathbf{p_B}^{\mathbf{0}}$
- Apply iteratively:

$$p_A^{t+1}(i) = \frac{1}{Z^t} p_A^t(i)\left(1 + \epsilon \cdot \mathbb{E}_{j \sim p_B^t}[r(i,j) | i]\right)$$

probability of player **A** playing
action **i** following the strategy $\mathbf{p_A}^{\mathbf{t+1}}$

MWUA for games

Given reward function **r** → find the optimal strategies $\mathbf{p}_A$ and $\mathbf{p}_B$

- We start with any strategies $\mathbf{p}_A^0$ and $\mathbf{p}_B^0$
- Apply iteratively:

$$p_A^{t+1}(i) = \frac{1}{Z^t} \underset{\uparrow}{p_A^t(i)} \left(1 + \epsilon \cdot \mathbb{E}_{j \sim p_B^t}[r(i,j) \mid i]\right)$$

probability of player **A** playing
action **i** following the strategy $\mathbf{p}_A^t$

MWUA for games

Given reward function **r** → find the optimal strategies $\mathbf{p_A}$ and $\mathbf{p_B}$

- We start with any strategies $\mathbf{p_A}^0$ and $\mathbf{p_B}^0$
- Apply iteratively:

$$p_A^{t+1}(i) = \frac{1}{Z^t} p_A^t(i)\left(1 + \epsilon \cdot \mathbb{E}_{j\sim p_B^t}[r(i,j)]\right)$$

$\uparrow$

multiplicative update

MWUA for games

Given reward function **r** → find the optimal strategies $\mathbf{p_A}$ and $\mathbf{p_B}$

- We start with any strategies $\mathbf{p_A}^{\mathbf{0}}$ and $\mathbf{p_B}^{\mathbf{0}}$
- Apply iteratively:

$$p_A^{t+1}(i) = \frac{1}{Z^t} p_A^t(i)\left(1 + \epsilon \cdot \underline{\mathbb{E}_{j\sim p_B^t}[r(i,j)|i]}\right)$$

expected reward if player **A** plays action **i**
and player **B** follows the strategy $\mathbf{p_B}^{\mathbf{t}}$

MWUA for games

Given reward function **r** → find the optimal strategies $\mathbf{p_A}$ and $\mathbf{p_B}$

- We start with any strategies $\mathbf{p_A^0}$ and $\mathbf{p_B^0}$
- Apply iteratively:

$$p_A^{t+1}(i) = \frac{1}{Z^t} p_A^t(i)(1 + \epsilon \cdot \mathbb{E}_{j \sim p_B^t}[r(i,j)|i])$$

small positive **«learning rate»**

MWUA for games

Given reward function **r** → find the optimal strategies $\mathbf{p}_\mathbf{A}$ and $\mathbf{p}_\mathbf{B}$

- We start with any strategies $\mathbf{p}_\mathbf{A}^\mathbf{0}$ and $\mathbf{p}_\mathbf{B}^\mathbf{0}$
- Apply iteratively:

$$p_A^{t+1}(i) = \frac{1}{Z^t} p_A^t(i)(1 + \epsilon \cdot \mathbb{E}_{j \sim p_B^t}[r(i,j)|i])$$

normalization constant such that $\mathbf{p}_\mathbf{A}^\mathbf{t+1}$ is a probability distribution

MWUA for games

$$p_A^{t+1}(i) = \frac{1}{Z^t} p_A^t(i)\left(1 + \epsilon \cdot \mathbb{E}_{j \sim p_B^t}[r(i,j) | i]\right)$$

Theorem

p_A^t and p_B^t converge to a nash equilibrium.

But why is this relevant for evolution?

Evolution

$$p_{ij}^{t+1} = \frac{w_{ij}}{Z^t}\left((1-r)p_{ij}^t + r\left(p_{i1}^t + p_{i2}^t\right)\left(p_{1j}^t + p_{2j}^t\right)\right)$$

MWUA for coordination games

$$p_A^{t+1}(i) = \frac{1}{Z^t} p_A^t(i)\left(1 + \epsilon \cdot \mathbb{E}_{j\sim p_B^t}[r(i,j)|i]\right)$$

Not quite there!

Theoretical Biology

Modelling Evolution (again)

Weak Selection

$$\forall i, j.\ w_{ij} \in [1 - s, 1 + s]$$

for a small **selection strength** $s > 0$

$$\Rightarrow w_{ij} = 1 + \mathrm{s} \cdot \Delta_{\mathrm{ij}},\ \ \Delta_{\mathrm{ij}} \in [-1, 1]$$

Weak Selection

assumption of weak selection

$$w_{ij} = 1 + \mathrm{s} \cdot \Delta_{\mathrm{ij}}, \qquad \Delta_{\mathrm{ij}} \in [-1, 1]$$

allele frequencies

$$p_A^{t+1}(i) \approx \frac{1}{Z^t} p_A^t(i)(1 + s \cdot \mathbb{E}_{j \sim p_B^t}[\Delta_{ij} | i])$$

probability of **allele i**
at **gene A** at **time t+1**

Evolution

$$p_A^{t+1}(i) \approx \frac{1}{Z^t} p_A^t(i)(1 + s \cdot \mathbb{E}_{j \sim p_B^t}[\Delta_{ij}|i])$$

MWUA for coordination games

$$p_A^{t+1}(i) = \frac{1}{Z^t} p_A^t(i)(1 + \epsilon \cdot \mathbb{E}_{j \sim p_B^t}[r(i,j)|i])$$

Looks similar?

$$p_A^{t+1}(i) \approx \frac{1}{Z^t} p_A^t(i)\left(1 + s \cdot \mathbb{E}_{j\sim p_B^t}\left[\Delta_{ij} \middle| i\right]\right)$$

frequency of **allele i** at **gene A** at **time t+1**

probability of **action i** by **player A** in **iteration t+1**

selection strength s

learning rate ϵ

expected **differential fitness** of genotypes with **allele i** at **gene A** at **time t**

expected **reward** upon **action i** by **player A** at **iteration t**

$$p_A^{t+1}(i) = \frac{1}{Z^t} p_A^t(i)(1 + \epsilon \cdot \mathbb{E}_{j\sim p_B^t}[r(i,j)|i])$$

Evolution implements MWUA

- **Evolution «implements» the MWU-algorithm**

- Genes are the players
- The alleles are the actions
- The allele frequencies are the strategies
- The reward is the differential fitness

What we learned

- Evolution under weak selection has the same behavior as the MWUA for coordination games
- Until convergence, there is a trade-off between fitness and diversity

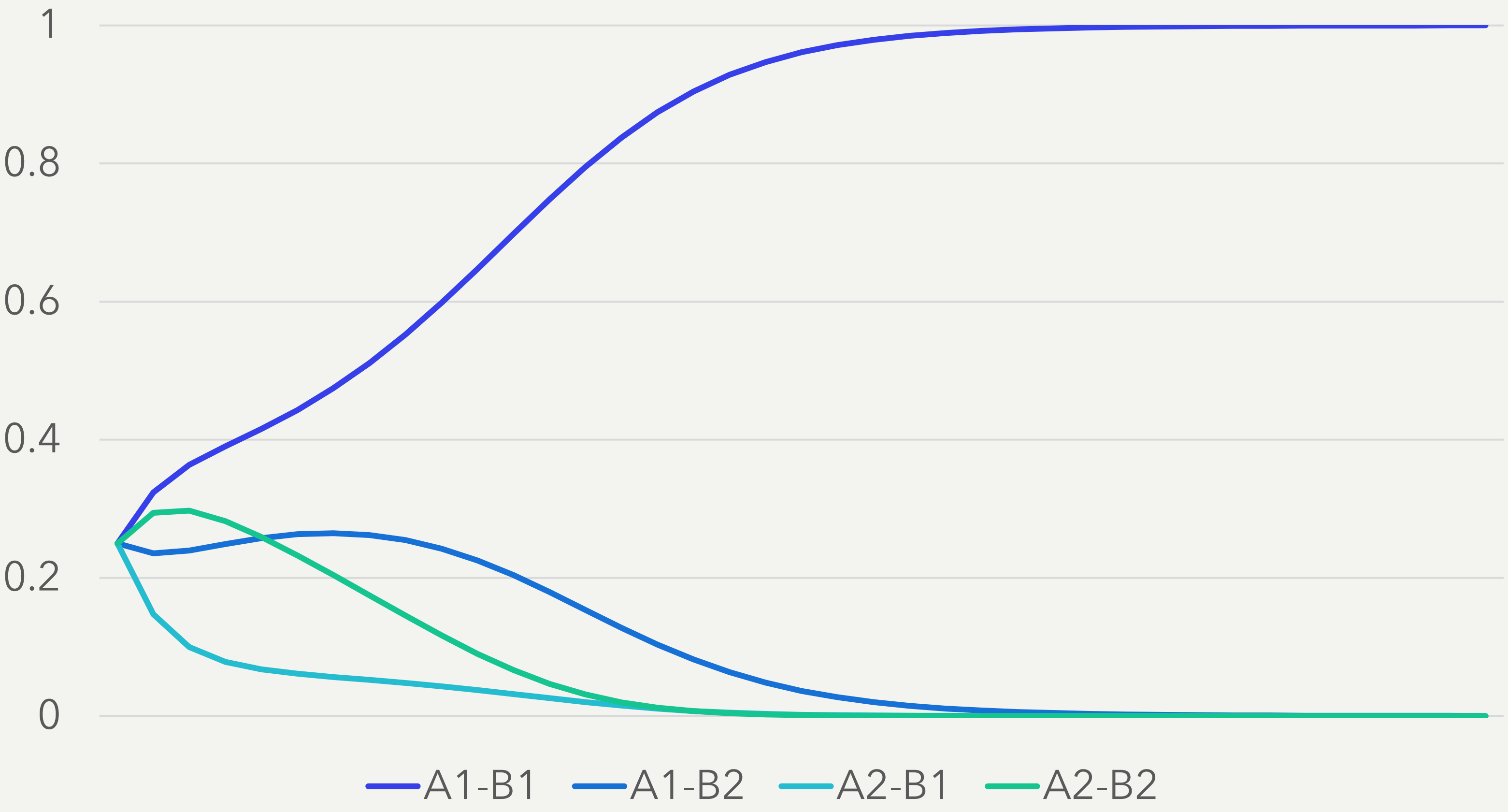
- Link between **Theoretical Biology**, **Game Theory**, **Theoretical Computer Science**, and **Information Theory!**

Thank you for your attention!

Information Theory

Source of Diversity

w_{ij}	B1	B2
A1	1.1	0.8
A2	0.5	1.0



$$p_{ij}^{t+1} = \frac{w_{ij}}{Z^t}((1-r)p_{ij}^t + r\left(p_{i1}^t + p_{i2}^t\right)\left(p_{1j}^t + p_{2j}^t\right))$$

There's more!

- Weak selection maximizes the following quantity:

$$x_A^t(1)\sum_{\tau=0}^{t}\mathbb{E}_{j\sim x_B^\tau}[\Delta_{1j}] + x_A^t(2)\sum_{\tau=0}^{t}\mathbb{E}_{j\sim x_B^\tau}[\Delta_{2j}] + \frac{1}{s}H(x_A^t)$$

cumulative expected differential fitness

entropy of the allele distribution
(high entropy equals high uncertainty)

There's more!

- Allele frequencies will eventually collapse
- Until collapse, **allele distributions with high uncertainty are preferred**

- This possibly explains the huge diversity in nature:
- **Trade-off between fitness and diversity until convergence!**

Weak Selection

$$\forall i, j.\ w_{ij} \in [1 - s, 1 + s]$$

for a small **selection strength** $s > 0$

$$\Rightarrow w_{ij} = 1 + \mathrm{s} \cdot \Delta_{\mathrm{ij}},\ \ \Delta_{\mathrm{ij}} \in [-1, 1]$$

Theorem (Nagylaki)

Under weak selection, the frequency p_{ij}^t of a genotype **Ai-Bj** can be approximated as a product of the marginal frequencies of the two alleles:

$$p_{ij}^t \approx \left(p_{i1}^t + p_{i2}^t\right) \cdot \left(p_{1j}^t + p_{2j}^t\right) = p_A^t(i) \cdot p_B^t(j)$$

The error of this approximation is in $O(s)$.

probability of **allele i**
at **gene A** at **time t**

$$w_{ij} \in [1-s, 1+s]$$

$$p_{ij}^t \approx \left(p_{i1}^t + p_{i2}^t\right) \cdot \left(p_{1j}^t + p_{2j}^t\right) = p_A^t(i) \cdot p_B^t(j)$$

Weak Selection

$$p_{ij}^{t+1} = \frac{w_{ij}}{Z^t}\left((1-r)p_{ij}^t + r\left(p_{i1}^t + p_{i2}^t\right)\left(p_{1j}^t + p_{2j}^t\right)\right)$$

$$\approx \frac{w_{ij}}{Z^t}\left((1-r) \cdot p_A^t(i) \cdot p_B^t(j) + r \cdot p_A^t(i) \cdot p_B^t(j)\right)$$

$$= \frac{w_{ij}}{Z^t} \cdot p_A^t(i) \cdot p_B^t(j)$$

$$= \frac{1 + s \cdot \Delta_{ij}}{Z^t} \cdot p_A^t(i) \cdot p_B^t(j)$$

($\Delta_{ij} \in [-1, 1]$ = **differential fitness** of genotype **Ai-Bj)**

$$p_{ij}^{t+1} \approx \frac{1 + s \cdot \Delta_{ij}}{Z^t} \cdot x_A^t(i) \cdot x_B^t(j)$$

Allele frequencies

$$p_A^{t+1}(i) = p_{i1}^{t+1} + p_{i2}^{t+1}$$

$$\approx \frac{1 + s \cdot \Delta_{i1}}{Z^t} \cdot p_A^t(i) \cdot p_B^t(1) + \frac{1 + s \cdot \Delta_{i2}}{Z^t} \cdot p_A^t(i) \cdot p_B^t(2)$$

$$= \frac{1}{Z^t} \cdot p_A^t(i) \cdot (p_B^t(1) + p_B^t(2) + s \cdot (\Delta_{i1} p_B^t(1) + \Delta_{i2} p_B^t(2)))$$

$$= \frac{1}{Z^t} p_A^t(i)(1 + s \cdot \mathbb{E}_{j \sim p_B^t}[\Delta_{ij} | i])$$

($\mathbb{E}_{j \sim p_B^t}[\Delta_{ij} | i] =$ **expected differential fitness** of a genotype with allele i for gene A at time t)

Weak Selection

assumption of weak selection

Nagylaki's theorem

$$w_{ij} \in [1-s, 1+s]$$

$$p_{ij}^{t} \approx p_{A}^{t}(i) \cdot p_{B}^{t}(j)$$

allele frequencies

$$p_A^{t+1}(i) \approx \frac{1}{Z^t} p_A^t(i)(1 + s \cdot \mathbb{E}_{j \sim p_B^t}[\Delta_{ij} | i])$$

$$p_B^{t+1}(j) \approx \frac{1}{Z^t} p_B^t(j)(1 + s \cdot \mathbb{E}_{i \sim p_A^t}[\Delta_{ij} | j])$$